# Weyl fields on one-sided bounded spacetimes

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## Introduction

Given a foliation of the spacetime $(\mathcal{M}, \mathrm{g})$ by space-like hypersurfaces $(\mathcal{H}_t)_t$, we derive new energy inequalities based on the observation that certain one-sided bounds on the geometry of this foliation are sufficient to control the Bel–Robinson energy of Weyl fields. The general purpose is to derive estimates that depend less on norms of geometric quantities and more directly on properties such as, for example, the eigenvalue spectrum of these geometric quantities.

By taking advantage of algebraic properties we identify minimal geometric conditions that are necessary and sufficient to derive energy inequalities for Weyl fields on curved spacetimes. In [1] we demonstrate that our methods may be applied to other classes of tensorial wave equations as well (Maxwell fields and Yang–Mills fields).

## Properties of the Bel–Robinson tensor

We assume that the spacetime is Ricci-flat and analyse the Bel–Robinson energy associated to the spactime curvature $\mathrm{R}$. From the curvature tensor and its Hodge dual ${}^*\mathrm{R}$ we may construct the *electric and magnetic parts* with respect to the future-directed, time-like unit normal vector field $\mathrm{N}$ ($X$ and $Y$ are vector fields tangent to $\mathcal{H}_t$):

$$\mathrm{E}(X,Y) := \mathrm{g}(\mathrm{R}(X,\mathrm{N})\mathrm{N},Y), \quad \mathrm{H}(X,Y) := \mathrm{g}({}^*\mathrm{R}(X,\mathrm{N})\mathrm{N},Y).$$

Relative to an orthonormal frame $\{\mathrm{e}_0,\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3\}$ with $\mathrm{e}_0 = \mathrm{N}$, the relevant components of the *Bel–Robinson tensor* $Q_{\alpha\beta\gamma\delta} = Q[\mathrm{R}]_{\alpha\beta\gamma\delta} := \mathrm{R}_{\alpha\lambda\gamma\mu}\mathrm{R}_\beta{}^\lambda{}_\delta{}^\mu + {}^*\mathrm{R}_{\alpha\lambda\gamma\mu}{}^*\mathrm{R}_\beta{}^\lambda{}_\delta{}^\mu$ may be decomposed as follows:

$$Q_{0000} = |\mathrm{E}|^2 + |\mathrm{H}|^2, \quad Q_{i000} = 2(\mathrm{E}\wedge\mathrm{H})_i, \quad Q_{ij00} = -(\mathrm{E}\times\mathrm{E})_{ij} - (\mathrm{H}\times\mathrm{H})_{ij} + \frac{1}{3}(|\mathrm{E}|^2 + |\mathrm{H}|^2)g_{ij}.$$

On Ricci-flat spacetimes, the Bel–Robinson tensor has the divergence-free property $\mathrm{D}^\alpha Q_{\alpha\beta\gamma\delta} = 0$.

## Derivation of an optimal algebraic condition

Let $\pi = \mathcal{L}_\mathrm{N}\mathrm{g}$ denote the *deformation tensor* of the foliation. It is straightforward to show that

$$\pi(\mathrm{N},\mathrm{N}) = 0, \quad \pi(\mathrm{N},X) = \nabla_X \log n, \quad \pi(X,Y) = -2\mathrm{k}(X,Y),$$

where $\nabla$ denotes the Levi-Civita connection of the slices $(\mathcal{H}_t, \mathrm{g}_t)$, $n$ the lapse function and $\mathrm{k}$ the second fundamental form of the foliation. It can be shown that the product $-\frac{1}{2}Q_{\alpha\beta00}n\pi^{\alpha\beta}$ can be controlled by the eigenvalues of a symmetric $6\times 6$ matrix $\Pi$ depending on $n$, $\pi$ and $\mathrm{k}$. More precisely, the largest eigenvalue $\rho(n\pi)$, which is the largest root of the characteristic polynomial of $\Pi$, can be used to derive

$$-\frac{1}{2}Q_{\alpha\beta00}n\pi^{\alpha\beta} \leq \rho(n\pi)Q_{0000}.$$

The value of $\rho(n\pi)$ can be estimated from the eigenvalues of $\mathrm{k}$ and the lapse $n$ together with $|\nabla n|$.

## Bel–Robinson energy inequality under a one-sided bound

The *total Bel–Robinson energy* at time $t$ is defined by

$$\mathcal{Q}[\mathrm{R}]_{\mathcal{H}_t} := \int_{\mathcal{H}_t} Q[\mathrm{R}](\mathrm{N},\mathrm{N},\mathrm{N},\mathrm{N})dV_{\mathrm{g}_t}.$$

By applying Stokes' Theorem to the vector field $Q_{\alpha\beta\gamma\delta}N^\beta N^\gamma N^\delta$ on the manifold with boundary $\mathcal{M}_{[t_1,t_2]} = \bigcup_{t\in[t_1,t_2]}\mathcal{H}_t$, as well as the above algebraic inequality and Gronwall's estimate, we deduce the following result.

### Theorem

*Let $I$ be an interval on the real line $\mathbb{R}$. Given any vacuum Einstein spacetime endowed with a foliation $(\mathcal{H}_t)_{t\in I}$ with lapse function $n$ and deformation tensor $\pi$, one has*

$$\mathcal{Q}[\mathrm{R}]_{\mathcal{H}_{t_2}} \leq e^{3K_{n\pi}(t_1,t_2)}\mathcal{Q}[\mathrm{R}]_{\mathcal{H}_{t_1}}, \qquad t_1 \leq t_2,$$

*for all $t_1, t_2 \in I$, where*

$$K_{n\pi}(t_1,t_2) := \int_{t_1}^{t_2} \sup_{\mathcal{H}_t} \rho(n\pi)dt.$$

## Generalization to Weyl fields

Consider a fixed Lorentzian manifold $(\mathcal{M}^{3+1},\mathrm{g})$. A Weyl field is a $(0,4)$-tensor field $W_{\alpha\beta\gamma\delta}$ that has the same symmetries as the Riemann tensor and, in addition, is trace-free. The Bel–Robinson tensor $Q[\mathrm{W}]$ and total Bel–Robinson energy $\mathcal{Q}[\mathrm{W}]_{\mathcal{H}_t}$ may be defined as above. In general, $\mathrm{div}(Q[\mathrm{W}]) \neq 0$, thus—with the same notation as above—, we deduce that any Weyl field $\mathrm{W}$ defined on $\mathcal{M}^{3+1}$ satisfies the energy inequality

$$\mathcal{Q}[\mathrm{W}]_{\mathcal{H}_{t_2}} \leq \mathcal{Q}[\mathrm{W}]_{\mathcal{H}_{t_1}}e^{3K_{n\pi}(t_1,t_2)} - \int_{t_1}^{t_2} e^{3K_{n\pi}(t_1,t_2)}\int_{\mathcal{H}_2} n(\mathrm{div}(Q[\mathrm{W}]))_{000}dV_\mathrm{g}dt, \qquad t_1 \leq t_2.$$

## References

[1] Burtscher, A.Y., Grant, J.D.E., LeFloch, P.G., New Energy Inequalities for Tensorial Wave Equations on Spacetimes that Satisfy a One-Sided Bound, *Comm. Partial Differential Equations* 37 (2012), 1596–1619.