Self-gravitating collapse of compressible matter under spherical symmetry

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Outline

1. Introduction
   - Gravitational collapse
   - Some known results
2. Trapped surface formation for perfect fluids

Theory of gravitational collapse

Goal

study of the `formation of black holes and singularities`

- for general asymptotically flat initial conditions,
- where no symmetry conditions are imposed.

physically interesting problems:

- initial conditions that are arbitrarily `far from` already `containing closed trapped surfaces`

Formation of trapped surfaces

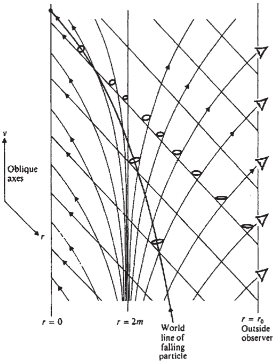
Trapped surface

A trapped surface is a spacelike surface with `decreasing area` in the direction of the null normals.

Example: **Schwarzschild metric** in Eddington–Finkelstein coordinates:

$$g = -\left(1 - \frac{2m}{r}\right) dv^2 + 2\, dv\, dr + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

- $r < 2m$: trapped surfaces
- $r = 2m$: event horizon



Some known results

Einstein–scalar field in spherical symmetry
(Christodoulou, 1990s)

- collapse of matter and formation of black holes
- formation of (instable) naked singularities

Einstein–Vlasov in spherical symmetry

- formation of trapped surfaces
  (Rendall 1992; Andréasson, Rein 2010)

Vacuum Einstein

- formation of trapped surfaces
  (Christodoulou 2008; Klainerman, Luc, Rodnianski 2013)

Outline

1. Introduction

2. Trapped surface formation for perfect fluids
   - Compressible matter
   - Reduced Einstein–Euler system
   - Existence of solutions and formation of trapped surfaces

Compressible matter

Energy-momentum tensor

$$T^{\alpha\beta} = (\mu + p)u^\alpha u^\beta + p g^{\alpha\beta}$$

- μ mass-energy density
- $p(\mu) = k^2\mu$ pressure with $k \in (0,1)$ speed of sound
- u^α velocity vector, normalized to $u^\alpha u_\alpha = -1$

Einstein–Euler equations

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} = 8\pi T_{\alpha\beta}$$
$$\nabla_\alpha T^{\alpha\beta} = 0$$

- satisfied in the `distributional sense`

Main result (outlook)

Theorem (B., LeFloch 2013)

There exists a class of `untrapped initial data` *prescribed on a hypersurface, that evolves to spherically symmetric Einstein–Euler spacetimes with bounded variation that contain* `trapped surfaces`.

Tasks:

- deal with low regularity
- find admissible initial data
- solve initial value problem and estimate time of existence
- show that trapping occurs during time of existence

Full Einstein–Euler system and regularity

Generalized Eddington–Finkelstein coordinates

$$g = -ab^2 dv^2 + 2b\,dv\,dr + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2)$$

with prescribed decay $\lim_{r\to 0} a(v,r) = \lim_{r\to 0} b(v,r) = 1$

Einstein equations for metric coefficients a, b

- `three first-order ODEs` (eq. between BV functions)
- `one second-order PDE` (in sense of distributions)

Euler equations for normalized fluid variables M, V

- system of `two coupled first-order PDEs`
- $M, V \in L^\infty([v_0, v_*], BV[0, r_* + \Delta]) \cap Lip([v_0, v_*], L^1[0, r_* + \Delta])$

Reduced Einstein–Euler system

Proposition (B., LeFloch 2013)

The full Einstein–Euler system is equivalent to a system of two first-order equations for the fluid,

$$\partial_v U + \partial_r F(U, a, b) = S(U, a, b),$$

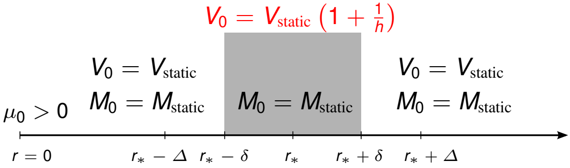
and two integral formulas for the geometry,

$$b(v, r) = \exp\left(4\pi(1 + k^2) \int_0^r M(v, s) s \, ds\right),$$

$$a(v, r) = 1 - \frac{4\pi(1+k^2)}{r} \int_0^r \frac{b(v,s)}{b(v,r)} M(v, s)(1 - 2\frac{1-k^2}{1+k^2} V(v, s)) s^2 \, ds.$$

1. Initial data sets

- ∃! smooth **static solutions** $(M, V, a, b)_{\text{static}}$ for any initial value $\mu_0 = \mu(0) > 0$ (based on Rendall, Schmidt 1991)
- static solutions satisfy desired `decay and regularity`
- fix radius $r_* > 0$ and $\Delta \in (0, r_*)$
- **compact perturbation** around $r_* > 0$, depending on $\delta \in (0, \Delta)$ and $h \leq 1$



1. Initial data sets

Proposition (B., LeFloch 2013)

There exist positive constants C_1, C_2 such that for $\frac{\delta}{h} \leq \frac{1}{C_1}$:

$$\begin{aligned} 0 < a_0(r) \leq a^{(0)}(r), \qquad & r \in [0, r_* + \Delta], \\ a_v(v_0, r) \leq -C_2 \frac{\delta}{h^3}, \qquad & r \in [r_* - \delta, r_* + \delta]. \end{aligned}$$

Thus initially we can choose the perturbation in a way that

- we have `untrapped surfaces`
- which are likely to become trapped.

2. Existence result for initial value problem

based on a **generalized random choice scheme** on a discretized grid (Groah, Temple 2004)

- consider `homogeneous Euler system`

$$\partial_v U + \partial_r F(U) = 0$$

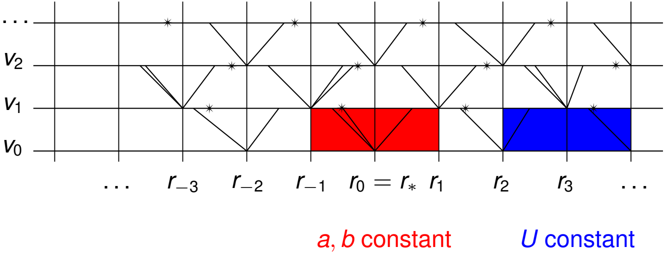
  on uniform geometric background (a, b constant)
  $\rightsquigarrow$ strictly hyperbolic, genuinely nonlinear system of conservation laws
- can solve `Riemann problem` explicitly for arbitrary initial data (U_L, U_R):

$$U(v_0, r) = \begin{cases} U_L & \text{if } r < r_0 \\ U_R & \text{if } r > r_0 \end{cases}$$

2. Existence result for initial value problem

- evolve solution of Riemann problem using `ODE system`

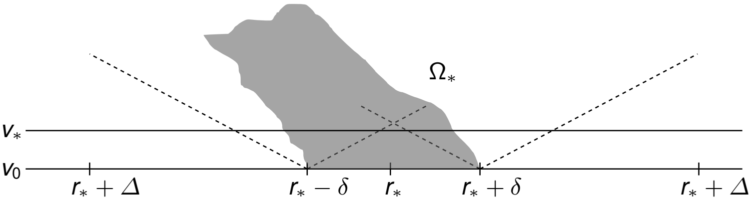
$$\partial_v W = \widetilde{S}(W, a, b) = S(\text{ource}) + \text{Geometry}(a, b)$$



$\Longrightarrow$ yields `approximate solutions` to the reduced Einstein–Euler equations in spherical symmetry

3. Control perturbation during evolution

- need control of `growth of solution` M, V, a, b
- `domain of dependence` Ω_*
- minimal `time of existence` v_*



3. Control perturbation during evolution

Perturbation property

holds if a solution (M, V, a, b) satisfies in the domain influenced by the perturbation for some constants $C, C_0, C_b, \Lambda > 0$, $\kappa > 1$:

$$\frac{1}{C_0} e^{-C\frac{v-v_0}{h^\kappa}} \leq M(v,r) \leq C_0 e^{C\frac{v-v_0}{h^\kappa}}$$

$$\frac{1}{C_0} e^{-C\frac{v-v_0}{h^\kappa}} \left(1+\frac{1}{h}\right) \leq -V(v,r) \leq C_0 e^{C\frac{v-v_0}{h^\kappa}} \left(1+\frac{1}{h}\right)$$

$$-\frac{1}{h} \leq a(v,r) \leq 1$$

$$1 \leq b(v,r) \leq C_b$$

$$-\frac{\Lambda}{h} \leq \lambda_i(v,r) \leq \frac{C_b}{2}.$$

Existence result

Theorem (B., LeFloch 2013)

Suppose `admissible initial data` (M_0, V_0, a_0, b_0) *are given. Then there exist constants* $\tau > 0, \kappa > 1$ *so that* `approximate solutions` *are well-defined on the interval* $[v_0, v_*]$ *with* $v_* = v_0 + \tau h^\kappa$*, and satisfy*

- *the perturbation property,*
- *a BV property in* r*, uniformly in* v*,*
- *an* L^1*-Lipschitz property in* v*.*

Consequently, a subsequence converges pointwise toward a limit (M, V, a, b) *which is a* ***bounded variation solution to the Einstein–Euler system in spherical symmetry*** *that satisfies the* `initial conditions` *and the* `perturbation property`*.*

Formation of trapped surfaces

Corollary (B., LeFloch 2013)

Fix $k \in (0,1)$ such that $\kappa < 2$. Suppose the initial data with $\frac{\delta}{h} = \frac{1}{C_1}$ additionally satisfy

$$8\pi r_* > e^3 \Lambda C_0^3 C_1.$$

*Then, if h is chosen sufficiently small, a **trapped surface** forms in the solution before time v_*, i.e. there exists $(v_\bullet, r_\bullet) \in \Omega_*$ such that $a(v_\bullet, r_\bullet) < 0$.*

- $\kappa < 2$ possible for k sufficiently small
- domain of dependence does not “close up” too soon
- initial data that satisfy this additional assumption do exist

Some open questions

- Can initial data be chosen asymptotically flat?
- What happens for *k* large (resp. $\kappa \geq 2$)?
- (How) do low regularity metrics effect the geometry?
- What about other symmetries?