Riemannian metrics of low regularity

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Outline

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Setting

- M ... connected smooth manifold
- $\mathbf{g}$ ... Riemannian metric

- $\mathcal{A}$ ... class of curves on M, $\gamma\colon [a,b] \to M$
- $L_{\mathbf{g}}(\gamma) = \int_a^b \|\gamma'(t)\|_{\mathbf{g}} dt$ ... arc-length

- $d_{\mathbf{g}}(p,q) = \inf\{L_{\mathbf{g}}(\gamma) \mid \gamma(a) = p, \gamma(b) = q\}$ ... distance fct.
- $L_{d_{\mathbf{g}}}(\gamma) = \sup_{\text{part.of}[a,b]} \sum_{i=1}^n d_{\mathbf{g}}(\gamma(t_{i-1}), \gamma(t_i))$ ... induced length

Smooth Riemannian metrics

Suppose $\mathbf{g}$ is a `smooth` Riemannian metric. Well-known:

- $(M, \mathcal{A}_{\mathcal{C}^1}, L_{\mathbf{g}})$ defines a `length structure` on M
- $d_{\mathbf{g}}$ is a metric on M that induces the manifold topology
- $(M, d_{\mathbf{g}})$ is a `length space`
- $d_{\mathbf{g}}$ and $d_{\mathbf{h}}$ are equivalent on compact sets

Moreover:

- solutions to the `geodesic` equation exist, are locally unique and length-minimizing
- the `exponential map` is a local diffeomorphism
- $L_{\mathbf{g}} = L_{d_{\mathbf{g}}}$ on $\mathcal{A}_{\mathcal{C}^1}$

Questions in the low regularity situation

What regularity of $\mathbf{g}$ and $\mathcal{A}$ is needed to obtain that ...

1. $d_{\mathbf{g}}$ is a `metric` on M (that induces the manifold topology)?
2. $d_{\mathbf{g}}$ and $d_{\mathbf{h}}$ are equivalent on compact sets?
3. solutions to the `geodesic` equation exist and are locally unique?
4. $L_{\mathbf{g}} = L_{d_{\mathbf{g}}}$ on $\mathcal{A}$?

Motivation to study the low-regularity situation

- Berestovskii, Nikolaev: Alexandrov spaces with two-sided curvature bounds are manifolds with $\mathcal{C}^{1,\alpha}$ ($0 < \alpha < 1$) Riemannian metrics
- Hartman (1950): local uniqueness of solutions to the geodesic equation does in general not hold for $\mathcal{C}^{1,\alpha}$ Riemannian metrics
- Kunzinger et al., Minguzzi (2013): for $\mathcal{C}^{1,1}$ (pseudo-)Riemannian metrics, the exponential map is a lipeomorphism
- Otsu–Shioya (1994): Alexandrov spaces with curvature bounded below inherit a $\mathcal{C}^0$-Riemannian structure (everywhere apart from a set of singular points)

Outline

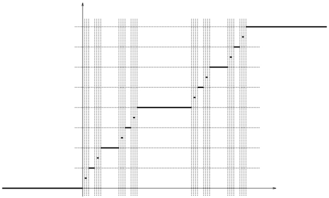
Differentiability a.e. is not enough

$M = \mathbb{R}$:

If γ is a BV curve, then it is differentiable a.e. and

$$L_{\mathbf{g}}(\gamma) \leq L_{d_{\mathbf{g}}}(\gamma),$$

with equality if and only if γ is absolutely continuous.



Cantor function

Absolute continuity on manifolds

Use a definition that only makes use of the differentiable structure and extends the notion of absolute continuity from $\mathbb{R}^n$.

Definition (AC curves, $\mathcal{A}_{\text{ac}}$)

A path $\gamma\colon I \to M$ is called `absolutely continuous` on M if for any chart (u, U) of M the composition

$$u \circ \gamma\colon \gamma^{-1}(\gamma(I) \cap U) \to u(U) \subseteq \mathbb{R}^n$$

is locally absolutely continuous.

Topology on AC curves

Definition (Variational metric on $\mathcal{A}_{\text{ac}}$)

For absolutely continuous paths $\gamma, \sigma \colon I \to M$ define

$$D_{\text{ac}}(\gamma,\sigma) = \sup_{t\in I} d_{\mathbf{g}}(\gamma(t),\sigma(t)) + \int_I \big| \|\gamma'(t)\|_{\mathbf{g}} - \|\sigma'(t)\|_{\mathbf{g}} \big| \, dt.$$

- $(M, d_{\mathbf{g}})$ is a metric space $\Rightarrow$ $(\mathcal{A}_{\text{ac}}(M), D_{\text{ac}})$ is a metric space
- $(M, d_{\mathbf{g}})$ is complete $\Rightarrow$ $(\mathcal{A}_{\text{ac}}(M), D_{\text{ac}})$ is complete
- $L_{\mathbf{g}} \colon \mathcal{A}_{\text{ac}} \to \mathbb{R}$ is Lipschitz continuous w.r.t. D_{ac}

AC vs. piecewise smooth curves

Theorem (Dense)

The class of p.w. smooth curves $\mathcal{A}_\infty$ is dense in the class of absolutely continuous curves $\mathcal{A}_{\mathrm{ac}}$ w.r.t. the `variational topology`*.*

Sketch of proof.

- locally approximate AC curves by smooth curves
- join up endpoints in a suitable way □

Corollary

- $d_{\mathbf{g},\mathcal{A}_\infty} = d_{\mathbf{g},\mathcal{A}_{\mathrm{ac}}}$
- $L_{d_{\mathbf{g}}} : t \mapsto L_{d_{\mathbf{g}}}(\gamma|_{[0,t]})$ *is AC for* $\gamma \in \mathcal{A}_{\mathrm{ac}}$.

Corollary (Equivalent notions of absolute continuity)

Let M be a connected smooth manifold and $\mathbf{g}$ a Riemannian metric s.t. $d_{\mathbf{g}}$ is a metric. Then the following are equivalent:

1. $\gamma\colon I \to M$ *is in* $\mathcal{A}_{\mathrm{ac}}$.
2. $\forall \varepsilon > 0 \exists \delta > 0$ *such that for any selection of disjoint intervals* $[a_i, b_i]_{i=1}^n \subseteq I$ *with* $\sum_{i=1}^n |b_i - a_i| < \delta$ *we have* $\sum_{i=1}^n d_{\mathbf{g}}(\gamma(a_i), \gamma(b_i)) < \varepsilon$.
3. $\exists f \in L^1(I)$ *such that* $\forall a < b$ *in* I: $d_{\mathbf{g}}(\gamma(a), \gamma(b)) \leq \int_a^b f(t)dt$.

Sketch of proof. (3 $\Rightarrow$ 2) $F(s) := \int_0^s f(t)dt$ is an AC function in $\mathbb{R}$
(2 $\Rightarrow$ 1) charts are diffeomorphisms (in particular Lipschitz)
(1 $\Rightarrow$ 3) use $f := \|\gamma'\|_{\mathbf{g}} \in L^1(I)$ and equality $d_{\mathbf{g},\mathcal{A}_\infty} = d_{\mathbf{g},\mathcal{A}_{\mathrm{ac}}}$ □

Length structure with AC curves

Theorem

Suppose **g** *is a smooth (or* $\mathcal{C}^2$*) Riemannian metric. Then* $L_{\mathbf{g}} = L_{d_{\mathbf{g}}}$ *on* $\mathcal{A}_{\mathrm{ac}}$*.*

Sketch of proof.

($L_{d_{\mathbf{g}}} \leq L_{\mathbf{g}}$) True by definition since $d_{\mathbf{g},\mathcal{A}_\infty} = d_{\mathbf{g},\mathcal{A}_{\mathrm{ac}}}$

($L_{d_{\mathbf{g}}} \geq L_{\mathbf{g}}$) Suppose $\gamma \in \mathcal{A}_{\mathrm{ac}}$ and t such that $\gamma'(t)$ exists (a.e.).

- exp is a local diffeomorphism $\Rightarrow |\dot{\gamma}(t)|_{d_{\mathbf{g}}} = \|\gamma'(t)\|_{\mathbf{g}}$
- & $(\leq) \Rightarrow \frac{d}{dt} L_{d_{\mathbf{g}}}(\gamma|_{[0,t]}) = \|\gamma'(t)\|_{\mathbf{g}}$
- & fundamental theorem of calculus applied to $L_{d_{\mathbf{g}}}$ (AC) implies the result. □

Outline

Riemannian metrics of low regularity

Lemma ($d_\mathbf{g}$ is a pseudo-metric)

Let $\mathbf{g}$ *be a Riemannian metric such that* $L_\mathbf{g}$ *is well-defined on* $\mathcal{A}_{\mathrm{ac}}$*. Then* $d_\mathbf{g}$ *is a pseudo-metric.*

- $d_\mathbf{g}(p,q) = d_\mathbf{g}(q,p)$
- $d_\mathbf{g}(p,q) \leq d_\mathbf{g}(p,r) + d_\mathbf{g}(r,q)$

Continuous Riemannian metrics

Theorem ($d_{\mathbf{g}}$ is a metric)

If $\mathbf{g}$ *is a* `continuous` *Riemannian metric, then*

1. $(M, \mathcal{A}_{\mathrm{ac}}, L_{\mathbf{g}})$ *is an admissible length structure on* M*,*
2. $d_{\mathbf{g}}$ *induces the manifold topology.*

In particular, $(M, d_{\mathbf{g}})$ *is a* `length space`*.*

Here, $(M, \mathcal{A}_{\mathrm{ac}}, L_{\mathbf{g}})$ is called an `admissible` length structure if

- $L_{\mathbf{g}}$ is additive, continuous on segments and invariant under reparametrizations
- $(M, \mathcal{A}_{\mathrm{ac}}, L_{\mathbf{g}})$ agrees with the `topology` on M, i.e. each point p has a neighborhood U such that the lengths of paths connecting p with $U^{\complement}$ is bounded away from 0.

Proof of (1)

Let p be a point on M, (u, U) a chart neighborhood of p such that $K := \overline{B_r(0)} \subseteq u(U)$ a compact set in $\mathbb{R}^n$.

- $e_U := \delta_{ij} dx^i \otimes dx^j$ . . . Euclidean metric w.r.t. U
- Let $0 < \eta_1 \leq \ldots \leq \eta_n$ be the eigenvalues and v_i be the eigenvectors of the symmetric tensor $e_U^{-1} \circ \mathbf{g}$, i.e.

$$\mathbf{g}(v_i, .) = \eta_i e_U(v_i, .).$$

- For $\lambda_0 = \min_{q \in K} \sqrt{n\eta_1(q)} > 0, \mu_0 = \max_{q \in K} \sqrt{n\eta_n(q)} < \infty$ and $q \in K, v \in T_qM$:

$$\lambda_0 \|v\|_{e_U} \leq \|v\|_{\mathbf{g}} \leq \mu_0 \|v\|_{e_U}. \qquad (*)$$

Proof of (1) continued

- Let $\gamma\colon I \to M$ be a path connecting p with a point in $M \setminus K^o$.
- Choose $t_0 \in I$ such that $\gamma(t_0) \in \partial K \cap \gamma(I)$. Then $(*) \Rightarrow$

$$
\begin{aligned}
0 < \lambda_0 r = \lambda_0 |(u \circ \gamma)(t_0)| &\leq \int_0^{t_0} \lambda_0 \|\gamma'(t)\|_{e_U} dt \\
&\leq \int_0^{t_0} \|\gamma'(t)\|_{\mathbf{g}} dt \leq L_{\mathbf{g}}(\gamma),
\end{aligned}
$$

i.e. the lengths of paths connecting p with $M \setminus K^o$ is bounded away from 0.

Proof of (2)

- distinguish two different ways of connecting p with $q \in K$
- $(*) \Rightarrow d_{\mathbf{g}}$ is locally Euclidean, i.e.

$$\lambda_0 d_{e_U}(p,q) \leq d_{\mathbf{g}}(p,q) \leq \mu_0 d_{e_U}(p,q),$$

and hence induces the manifold topology. □

Distance functions are equivalent

Theorem

Let $\mathbf{g}, \mathbf{h}$ be two continuous Riemannian metrics on M. On every compact set K in M there exists a constant $C > 0$ such that

$$\frac{1}{C} d_{\mathbf{h}}(p,q) \leq d_{\mathbf{g}}(p,q) \leq C d_{\mathbf{h}}(p,q), \qquad p,q \in K.$$

This is not true for arbitrary (compact) metric spaces:

$$d_1(x,y) = |x-y|, \qquad d_2(x,y) = \sqrt{|x-y|}$$

induce the same topology on $[0,1]$, but $\frac{d_2(0,1/n)}{d_1(0,1/n)} \to \infty$.

Proof

- Suppose to the contrary that there exist $(p_n)_n, (q_n)_n \in K$ with
$$d_{\mathbf{g}}(p_n, q_n) > n\, d_{\mathbf{h}}(p_n, q_n)$$
- w.l.o.g. $p_n, q_n \to p \in K$
- M is locally compact and $d_{\mathbf{h}}$ induces the manifold topology $\Rightarrow \exists r_0 > 0$ such that $\overline{B^{\mathbf{h}}_{r_0}(p)} = \{q \in M | d_{\mathbf{h}}(p, q) \leq r_0\}$ is compact.
- choose $r := \frac{r_0}{4}$ and $x, y \in B^{\mathbf{h}}_r(p)$
- for all $\varepsilon \in (0, r)$ there exists a path γ_ε between x and y such that $L(\gamma_\varepsilon) < d_{\mathbf{h}}(x, y) + \varepsilon$

Proof continued

- γ_ε is mapped to $B^{\mathbf{h}}_{r_0}(p)$:

$$\begin{aligned} d_{\mathbf{h}}(p, \gamma_\varepsilon(t)) &\leq d_{\mathbf{h}}(p, x) + d_{\mathbf{h}}(x, \gamma_\varepsilon(t)) \\ &\leq r + L_{\mathbf{h}}(\gamma_\varepsilon) < r + 2r + r = r_0 \end{aligned}$$

- $\mathbf{g}, \mathbf{h}$ are continuous $\Rightarrow$ same inequalities $(*)$ as in previous proof hold on $\overline{B^{\mathbf{h}}_{r_0}(p)}$: for $C = \frac{\mu_0^{\mathbf{g}}}{\lambda_0^{\mathbf{h}}} > 0$,

$$d_{\mathbf{g}}(x, y) \leq L_{\mathbf{g}}(\gamma_\varepsilon) \leq C L_{\mathbf{h}}(\gamma_\varepsilon) < C d_{\mathbf{h}}(x, y) + C\varepsilon$$

- for n sufficiently large therefore

$$d_{\mathbf{g}}(p_n, q_n) \leq C d_{\mathbf{h}}(p_n, q_n),$$

a contradiction. □

Arc-length vs. induced length

Theorem

Let $\mathbf{g}$ be a continuous Riemannian metric. Then for any absolutely continuous curve on M,

$$L_{\mathbf{g}}(\gamma) = L_{d_{\mathbf{g}}}(\gamma).$$

The proof is obtained in two steps:

- $L_{d_{\mathbf{g}}} = \widetilde{L}$ (metric part)
- $\widetilde{L} = L_{\mathbf{g}}$ (Riemannian part)

From the analysis of metric spaces I

Definition

Let (X, d) be a metric space, $\gamma\colon I \to X$ a path. The `metric derivative` of γ is denoted by

$$|\dot\gamma|(t) := \lim_{\delta\to 0} \frac{d(\gamma(t+\delta), \gamma(t))}{|\delta|}$$

(whenever it exists).

If **g** is `smooth` $\Rightarrow$ exponential map is a diffeomorphism $\Rightarrow$

$$|\dot\gamma| = \left\| \frac{d}{d\delta}\Big|_0 \exp^{-1}_{\gamma(t)}(\gamma(t+\delta)) \right\|_{\mathbf{g}}$$
$$= \left\| (T_0 \exp_{\gamma(t)})^{-1}(\gamma'(t)) \right\|_{\mathbf{g}} = \|\gamma'(t)\|_{\mathbf{g}}$$

From the analysis of metric spaces II

- If γ is AC, then $|\dot{\gamma}|$ exists a.e., and $|\dot{\gamma}|$ is the `minimal` L^1 function satisfying the inequality in the "definition" of absolute continuity (cf. Ambrosio–Gigli–Savaré 2005)
- $\Rightarrow \int_I |\dot{\gamma}|(t)dt \leq L_{\mathbf{g}}(\gamma)$
- for Lipschitz/AC curves γ (cf. Ambrosio–Tilli 2004):

$$\widetilde{L}(\gamma) := \int_I |\dot{\gamma}|(t)dt = L_d(\gamma)$$

Uniform convergence

Lemma

Let $\mathbf{g}$ *be a continuous Riemannian metric. There exists a sequence of smooth Riemannian metrics* $(\mathbf{g}_n)_n$ *such that*

1. $\mathbf{g}_n \to \mathbf{g}$ *uniformly and*
2. $d_{\mathbf{g}_n} \to d_{\mathbf{g}}$ *uniformly on* M*.*

Proof.

- locally, on compact subsets K_p, approximate by convolution with mollifiers to get $(\mathbf{h}_n)_n$ such that

$$\frac{n-1}{n}\|v\|_{\mathbf{g}} \leq \|v\|_{\mathbf{h}_n^p} \leq \frac{n+1}{n}\|v\|_{\mathbf{g}}$$

Proof continued

- use a partition of unity $\{\alpha_p\}_{p\in M}$ subordinate to the cover $\{K_p^o\}_{p\in M}$ to define the approximations

$$\mathbf{g}_n := \sum_{p\in M} \alpha_p \mathbf{h}_n^p$$

satisfying the same estimates on M

- using the definition of $d_{\mathbf{g}}, d_{\mathbf{g}_n}$ one also has for $p, q \in M$

$$\frac{n-1}{n} d_{\mathbf{g}}(p,q) \leq d_{\mathbf{g}_n}(p,q) \leq \frac{n+1}{n} d_{\mathbf{g}}(p,q). \qquad \square$$

Derivatives

Lemma

Let $\mathbf{g}$ *be a continuous Riemannian metric on a manifold* M*. For any absolutely continuous path* $\gamma\colon I \to M$ *the following equality holds for a.e.* $t \in I$*:*

$$|\dot\gamma|(t) := \lim_{\delta\to 0} \frac{d_{\mathbf{g}}(\gamma(t+\delta),\gamma(t))}{|\delta|} = \|\gamma'(t)\|.$$

Proof. uniform convergence allows interchange of limits

$$|\dot\gamma|_{d_{\mathbf{g}}}(t) = \lim_{\delta\to 0}\lim_{n\to\infty} \frac{d_{\mathbf{g}_n}(\gamma(t+\delta),\gamma(t))}{|\delta|} = \lim_{n\to\infty} |\dot\gamma|_{d_{\mathbf{g}\,n}}(t)$$
$$= \lim_{n\to\infty} \|\gamma'(t)\|_{\mathbf{g}_n} = \|\gamma'(t)\|_{\mathbf{g}}. \qquad \square$$

Proof of Theorem

Let $\gamma\colon I \to M$ be an absolutely continuous curve. Then by the above

$$L_{d_{\mathbf{g}}} = \sup_{\text{part.of } I} \sum_{i=1}^{n} d_{\mathbf{g}}(\gamma(t_{i-1}), \gamma(t_i))$$
$$= \int_I |\dot\gamma|_{d_{\mathbf{g}}}(t)\,dt$$
$$= \int_I \|\gamma'(t)\|_{\mathbf{g}}\,dt = L_{\mathbf{g}}(\gamma). \quad \square$$

Summary

- arc-length definition is only meaningful for `absolutely continuous curves`
- below this regularity use `induced length`
- for absolutely continuous curves the metric/analytic `derivatives coincide`, and therefore also the different notions of `lengths`
- a `continuous Riemannian metric` is needed to obtain an admissible length structure that respects the topology
- and to find `uniform approximations` by smooth Riemannian metrics