On the mathematical framework in general relativity

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Outline

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Historical Overview

Classical mechanics (Newton)

- `static` geometric background
- not accurate on large-scale

Einstein's theory of relativity 1915

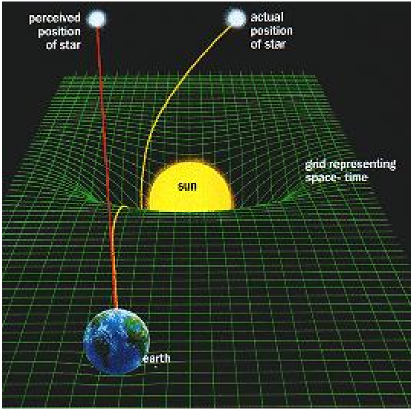
- `dynamic` "spacetime"
- describes astronomical observations accurately

Eddington's experiment (deflection of light)

`dynamic` spacetime

→ matter deforms the geometry of spacetime

→ the geometry of spacetimes determines how matter moves



General relativity

Key idea of general relativity

`Gravitation` is not a force but a `geometric` property of space and time.

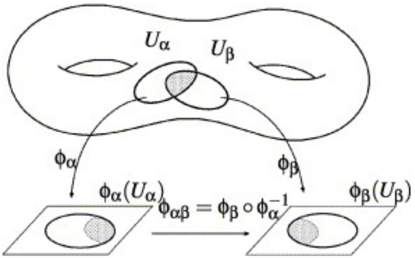
Description via the Einstein equations.

Outline

Manifolds

Manifolds $\mathcal{M}$ are the main object in differential geometry. They are equipped with

- topology
- local charts
- compatibility



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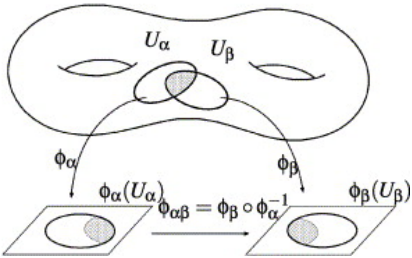

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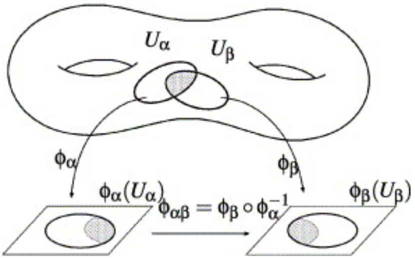


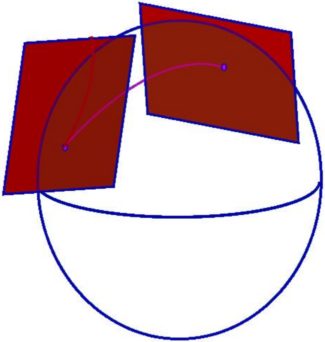
Examples: open subsets of $\mathbb{R}^n$, sphere ...

Tangent space

Let p be a point of the manifold $\mathcal{M}$.

A **tangent space** $T_p\mathcal{M}$ is a real vector space attached to p.

The **tangent bundle** $T\mathcal{M}$ is the disjoint union $\bigcup T_p\mathcal{M}$ of all tangent spaces.



Tensor fields

An (r, s) **tensor field** $A \in \mathcal{T}^r_s(\mathcal{M})$ on $\mathcal{M}$ defines a multilinear map

$$A(p)\colon \underbrace{T_p\mathcal{M}^* \times \ldots \times T_p\mathcal{M}^*}_{r-\text{times}} \times \underbrace{T_p\mathcal{M} \times \ldots \times T_p\mathcal{M}}_{s-\text{times}} \to \mathbb{R}$$

at each point $p \in \mathcal{M}$.

Examples: $\mathcal{T}^1_0(\mathcal{M})$ are vector fields, $\mathcal{T}^0_1(\mathcal{M})$ are 1-forms

Lorentzian metric

We smoothly assign to each point a scalar product on the tangent space:

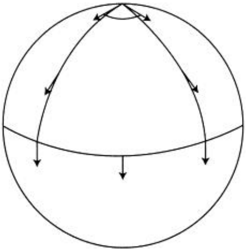
Lorentzian metric

A Lorentzian metric tensor **g** on a manifold $\mathcal{M}$ is a `symmetric, non-degenerate (0,2)-tensor field` on $\mathcal{M}$ with constant index 1.

Example on $\mathbb{R}^{n+1}$: $\langle v_p, w_p \rangle = -v^0 w^0 + \sum_{i=1}^{n} v^i w^i$

Curvature (short)

- For each Lorentzian manifold there exists the `unique Levi-Civita connection` ∇ (covariant derivative).
- `Parallel transport` is described in terms of covariant derivation.
- Failure of commutativity is a measure for `curvature`.



Curvature (exact)

Definition

Let $\mathcal{M}$ be a Lorentzian manifold with Levi-Civita connection ∇. The $(1,3)$-tensor field **R**, defined by

$$R_{XY}(Z) := \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

is called **Riemann curvature tensor.**

Ricci and scalar curvature

Contractions of Riemannian curvature yield simpler invariants:

Definition

Ricci curvature Ric is the C_3^1 contraction of **R**: $R_{ij} = \sum R_{ijm}^m$

Definition

Scalar curvature S is the contraction of **Ric**: $S = \sum g^{ij} R_{ij}$

Model

The Einstein equations give a `relation between the curvature` of spacetime `and the matter distribution` of the universe

- modeled by a 4-dimensional Lorentzian manifold $(\mathcal{M}, \mathbf{g})$
- gravitation is an effect of the curvature of $\mathcal{M}$
- matter distribution is given by the energy-momentum tensor $\mathbf{T}$

Adaption

Conservation of energy: $div\mathbf{T} = 0$
but: $div\mathbf{Ric} \neq 0$

Definition

The **Einstein tensor** of a spacetime is

$$\mathbf{G} = \mathbf{Ric} - \frac{1}{2} S\mathbf{g}.$$

now: $div\mathbf{G} = 0$

Einstein equations

Einstein equations

$$\mathbf{G} = 8\pi\mathbf{T}$$

General relativity is the study of the solutions of this system of equations – a `system of coupled nonlinear partial differential equations.`

Outline

Historical Overview

1915: Einstein introduced his equations

1952: Choquet-Bruhat proved that Einstein equations can be formulated as an `initial value problem` and showed `local existence of solutions`

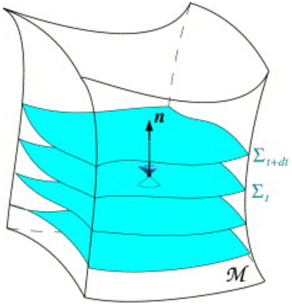
later: improvements on regularity

Foliation

Initial data should consist of

- a 3-dim. manifold Σ
- with a Riemannian metric **h**
- and a symmetric tensor field **K** on Σ

... but this is not enough!



Initial data constraints

need compatibility of the curvature in $\mathcal{M}$ and the curvature in Σ

Gauß and Codazzi equations + (vacuum) Einstein equations $\rightarrow$

$$S(h) = |\mathbf{K}|_h^2 - (tr_h\mathbf{K})^2$$
$$\nabla_j K^j{}_k - \nabla_k K^j{}_j = 0$$

Solving the constraint equations

The constraint equations can be `decomposed` in different ways and transformed to **elliptic equations** and solved.

→ get **initial data** (non-unique!)

Gauge fixing

By fixing coordinates, the Einstein equations can be simplified:

e.g. `harmonic coordinates` x^μ such that $\Box_{\mathbf{g}} x^\mu = 0$

$\rightarrow$ can write (vacuum) Einstein equations as a **system of quasilinear hyperbolic partial differential equations**

$$-\frac{1}{2}\sum_{\alpha,\beta} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + F_{\mu\nu}(\mathbf{g}, \partial\mathbf{g}) = 0$$

$\rightarrow$ existence result using standard theory

Local existence

Theorem

Suppose **h** *and* **K** *are smooth on* Σ *and the constraints are satisfied, then the initial value problem has a smooth solution in the neighborhood of* Σ*.*

generalizations: $\mathbf{h} \in H^s_{\mathrm{loc}}(\Sigma)$, $\mathbf{K} \in H^{s-1}_{\mathrm{loc}}(\Sigma)$ for $s > 2$ (Klainerman–Rodnianski 2005)

Outline

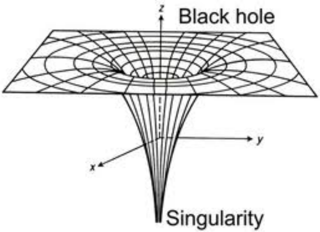
Global existence?

Can solutions be extended globally?

Intuitively: a singularity is a place where the `curvature of spacetime becomes infinite`

→ but these points are in fact missing from the solution

→ spacetime is incomplete in some sense



Singularity theorems

Theorem (Hawking and Penrose, 1970)

Spacetime $(\mathcal{M}, \mathbf{g})$ *is* `not` *timelike and null geodesically* `complete` *if:*

- $R_{\alpha\beta} V^{\alpha} V^{\beta} \geq 0$ *for every non-spacelike vector* $\mathbf{V}$
- *A generic condition for tangent vectors holds.*
- *There are no closed timelike curves.*
- *There exists at least one of the following: a compact achronal set without edges, a closed trapped surface, or a point p such that null geodesics from p are focussed by the matter or curvature and start to reconverge.*

Breakdown criteria

Theorem (Klainerman and Rodnianski, 2010)

Let $(\mathcal{M}, \mathbf{g})$ be a globally hyperbolic development of Σ foliated by the CMC level hypersurfaces of a time function $t < 0$, such that Σ corresponds to the level surface $t = t_0$. Assume that Σ satisfies the specific metric inequality. Then the first time $T < 0$, with respect to the t-foliation, of a breakdown is characterized by the condition

$$\limsup_{t \to T^-} \left(\|\mathbf{K}(t)\|_{L^\infty} + \|\nabla \log n(t)\|_{L^\infty} \right) = \infty.$$

More precisely the spacetime together with the foliation Σ_t can be extended beyond any value $T < 0$ for which the above value is finite.

Outline

Summary

- General relativity describes gravitation in terms of curvature.
- Solutions to the Einstein equations with appropriate initial data exist locally.
- Global behavior is determined by the formation of singularities.

For Further Reading

S. Hawking and Ellis.
The large scale structure of space-time.
Cambridge University Press, 1973.

B. O'Neill.
Semi-Riemannian Geometry (with applications to relativity).
Academic Press, 1983.

S. Klainerman and I. Rodnianski.
On the breakdown criterion in general relativity.
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