Energy inequalities under one-sided curvature bounds

Annegret Y. Burtscher

joint work with

James D.E. Grant and Philippe G. LeFloch

Faculty of Mathematics
University of Vienna

Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie (Paris 6)

Relativity Seminar Golm, 7 Feb 2013

Outline

Outline

Curved Spacetime

- $(\mathcal{M}^{3+1}, \mathbf{g})$ is a globally hyperbolic spacetime, Ricci-flat
- foliated by compact or asymptotically flat `hypersurfaces` $\mathcal{H}_t$, time parameter $t \in I$
- $\mathbf{N}$ future-directed time-like unit normal
- n lapse
- $\mathbf{k}$ second fundamental form
- orthonormal frame $\{\mathbf{e}_0 = \mathbf{N}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

Change of Metric in Normal Direction

Deformation tensor of the foliation

$$\pi = \mathcal{L}_{\mathbf{N}}\mathbf{g}$$

Components:

- $\pi_{00} = 0$
- $\pi_{0i} = \nabla_i \log n$
- $\pi_{ij} = -2k_{ij}$

Bel–Robinson Tensor

Bel–Robinson tensor

$$Q_{\alpha\beta\gamma\delta} = Q[\mathbf{R}]_{\alpha\beta\gamma\delta} := R_{\alpha\lambda\gamma\mu} R_{\beta}{}^{\lambda}{}_{\delta}{}^{\mu} + {}^{\star}R_{\alpha\lambda\gamma\mu}{}^{\star}R_{\beta}{}^{\lambda}{}_{\delta}{}^{\mu}$$

- can be decomposed using the `electric and magnetic parts` of the curvature tensor **R**, e.g.

$$Q_{0000} = E_{ij}E^{ij} + H_{ij}H^{ij} = |\mathbf{E}|^2 + |\mathbf{H}|^2$$

$$E_{ij} = R_{i0j0} \qquad H_{ij} = {}^{\star}R_{i0j0}$$

- is divergence-free on Ricci-flat spacetimes: $\mathbf{D}^{\alpha} Q_{\alpha\beta\gamma\delta} = 0$

Bel–Robinson Energy

Total Bel–Robinson energy at time t

$$\mathcal{Q}[\mathbf{R}]_{\mathcal{H}_t} := \int_{\mathcal{H}_t} Q[\mathbf{R}](\mathbf{N},\mathbf{N},\mathbf{N},\mathbf{N})\, dV_{\mathbf{g}_t}$$

`WANT:` control the Bel–Robinson energy at time t by its values on some initial slice assuming a `two-sided` bound on the deformation tensor one obtains

Theorem (e.g. Wang 2012)

Suppose $\int_{t_0}^{t_1} \|\pi\|_{L^\infty(\mathcal{H}_s)} ds = K < \infty$ *then there exists a constant* $C = C(K, t_1) > 0$ *such that for all* $t \in [t_0, t_1)$

$$\mathcal{Q}(t) \leq C\mathcal{Q}(t_0).$$

Outline

Main Result

Theorem (B., Grant, LeFloch 2012)

Given a vacuum spacetime endowed with a foliation $(\mathcal{H}_t)_{t\in I}$ with lapse n and second fundamental form $\mathbf{k}$, one has for $t_1 \leq t_2$, $t_1, t_2 \in I$,

$$\mathcal{Q}[\mathbf{R}]_{\mathcal{H}_{t_2}} \leq e^{3K_{n,\mathbf{k}}(t_1,t_2)} \mathcal{Q}[\mathbf{R}]_{\mathcal{H}_{t_1}},$$

where

$$K_{n,\mathbf{k}}(t_1, t_2) := \int_{t_1}^{t_2} \sup_{\mathcal{H}_t} \rho(n, \mathbf{k}) dt$$

and $\rho(n, \mathbf{k})$ is the largest eigenvalue of the symmetric 6×6 matrix $\Pi(n, \mathbf{k})$.

Sketch of Proof I

- express the Bel-Robinson tensor in terms of electric and magnetic parts of curvature to obtain

$$-\frac{1}{2}Q_{\alpha\beta00}n\pi^{\alpha\beta} = \mathrm{tr}\left(\begin{pmatrix}\mathbf{E} & \mathbf{H}\end{pmatrix}\Pi\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right)$$

  with a symmetric 6×6 matrix Π depending on n, $\mathbf{k}$ and π
- the matrix Π has 3 real-valued double eigenvalues $a_1 \leq a_2 \leq a_3 =: \rho(n, \mathbf{k})$
-

$$-\frac{1}{2}Q_{\alpha\beta00}n\pi^{\alpha\beta} \leq a_3\,\mathrm{tr}\left(\begin{pmatrix}\mathbf{E} & \mathbf{H}\end{pmatrix}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}\right) = \rho(n,\mathbf{k})Q_{0000}$$

Sketch of Proof II

- since the Bel–Robinson tensor is `divergence-free` we have $\mathbf{D}^\alpha Q_{\alpha 000} = \frac{3}{2} Q_{\alpha\beta 00}\pi^{\alpha\beta}$
- and Stokes' theorem implies on $\mathcal{M}_{[t_1,t_2]} = \bigcup_{t\in[t_1,t_2]} \mathcal{H}_t$:

$$\mathcal{Q}_{\mathcal{H}_{t_1}} - \mathcal{Q}_{\mathcal{H}_{t_2}} = \frac{3}{2}\int_{\mathcal{M}_{[t_1,t_2]}} Q_{\alpha\beta 00}\pi^{\alpha\beta} dV_{\mathbf{g}}$$

$$\geq -3\int_{t_1}^{t_2} \sup_{\mathcal{H}_s}(\rho(n,\mathbf{k})\mathcal{Q}_{\mathcal{H}_s} ds.$$

- Gronwall argument: $\mathcal{Q}_{\mathcal{H}_{t_2}} \leq e^{3\int_{t_1}^{t_2} \sup_{\mathcal{H}_t} \rho(n,\mathbf{k}) dt} \mathcal{Q}_{\mathcal{H}_{t_1}}$

Generalization to Weyl Fields

Weyl field

A `Weyl field` is a $(0,4)$-tensor field $W_{\alpha\beta\gamma\delta}$ that has the same symmetries as the Riemann tensor and is trace-free, i.e.

$$W_{\alpha\beta\gamma\delta} = W_{[\alpha\beta][\gamma\delta]} \quad W_{[\alpha\beta\gamma]\delta} = 0 \quad W^{\alpha}_{\beta\alpha\delta} = 0.$$

`Bel-Robinson tensor` and `energy`:

$$Q[\mathbf{W}]_{\alpha\beta\gamma\delta} := W_{\alpha\lambda\gamma\mu} W_{\beta}{}^{\lambda}{}_{\delta}{}^{\mu} + {}^{\star}W_{\alpha\lambda\gamma\mu} {}^{\star}W_{\beta}{}^{\lambda}{}_{\delta}{}^{\mu}$$

$$\mathcal{Q}[\mathbf{W}]_{\mathcal{H}_t} := \int_{\mathcal{H}_t} Q[\mathbf{W}](\mathbf{N},\mathbf{N},\mathbf{N},\mathbf{N}) dV_{\mathbf{g}_t}$$

Result on Weyl Fields

Theorem (B., Grant, LeFloch 2012)

Same notation as before. Any Weyl field **W** *defined on* $\mathcal{M}^{3+1}$ *satisfies*

$$\mathcal{Q}[\mathbf{W}]_{\mathcal{H}_{t_2}} \leq e^{3K_{n,\mathbf{k}}(t_1,t_2)}\mathcal{Q}[\mathbf{W}]_{\mathcal{H}_{t_1}} - \int_{t_1}^{t_2} e^{3K_{n,\mathbf{k}}(t,t_2)} \int_{\mathcal{H}_t} n(\mathrm{div}\, Q[\mathbf{W}])_{000}\, dV_{\mathbf{g}_t}$$

for $t_1 \leq t_2$, $t_1, t_2 \in I$.

Summary

- The growth of the `Bel-Robinson energy` can be controlled by the lapse and the second fundamental form of the foliation.
- Algebraic properties of the Bel–Robinson tensor show that `one-sided geometric bounds` are sufficient.
- Other observations
  - similar results can be obtained using Maxwell and Yang-Mills Fields
  - bounds of the deformation tensor play also a role in the formulation of breakdown criteria for the vacuum Einstein equations (Klainerman–Rodnianski 2010, Wang 2012)