Smooth regularization parameter in special Colombeau algebras

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Notation

Compare special Colombeau algebras with different parameterization:

Definition

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and X, Y manifolds:

- $\widetilde{\mathbb{K}}$, $\widetilde{\mathbb{K}}_0$, $\widetilde{\mathbb{K}}_\infty$ ... rings of **generalized numbers**
- $\mathcal{G}(X)$, $\mathcal{G}_0(X)$, $\mathcal{G}_\infty(X)$ ... **special Colombeau algebras** on X
- $\mathcal{G}[X, Y]$, $\mathcal{G}_0[X, Y]$, $\mathcal{G}_\infty[X, Y]$ ... c-bounded gen. fct. valued in Y

with usual/continuous/smooth dependence on $\varepsilon \in (0, 1]$.

Smooth dependence: advantages

Algebraic simplifications (Biagioni, Oberguggenberger):

- Polynomials with generalized coefficients only have *classical solutions* if at least continuous dependence on ε

Sheaf and embedding properties (Kunzinger, Steinbauer, Vickers):

- $\mathcal{G}_\infty[\_, Y]$ is a sheaf of sets.
- Embedding of *continuous functions*: There exists an embedding $\iota : \mathcal{C}(X, Y) \hookrightarrow \mathcal{G}_\infty[X, Y]$ such that ι is a sheaf morphism.

Prototypical result

Theorem

Let $\Psi : \mathcal{G}_\infty(X) \to \mathcal{G}_\infty(Y)$ be an algebra isomorphism. Then there $\exists!\, \psi \in \mathcal{G}_\infty[Y, X]$ invertible such that

$$\begin{aligned} \Psi(u) &= u \circ \psi & \forall u \in \mathcal{G}_\infty(X), \\ \Psi^{-1}(v) &= v \circ \psi^{-1} & \forall v \in \mathcal{G}_\infty(Y). \end{aligned}$$

Modified version of H. Vernaeve's result:

- algebra isomorphisms $\Psi : \mathcal{G}(X) \to \mathcal{G}(Y)$ are pullbacks by $\psi \in \mathcal{G}_{\mathrm{ld}}[Y, X]$ (locally defined c-bounded generalized function)

Sketch of proof

(1) Non-zero algebra homomorphisms $\varphi : \mathcal{G}_\infty(X) \to \widetilde{\mathbb{K}}_\infty$

identified with compactly supported generalized points in X:

$$\forall \varphi \, \exists! \, \tilde{p} \in \widetilde{X}_{c,\infty} : \; \varphi = \mathrm{ev}_{\tilde{p}}$$

(2) Algebra isomorphisms $\Psi : \mathcal{G}_\infty(X) \to \mathcal{G}_\infty(Y)$

For all $\tilde{q} \in \widetilde{Y}_{c,\infty}$ consider the algebra homom. $\mathrm{ev}_{\tilde{q}} \circ \Psi : \mathcal{G}_\infty(X) \to \widetilde{\mathbb{K}}_\infty$

$\rightsquigarrow \tilde{p} \in \widetilde{X}_{c,\infty}$ such that $\mathrm{ev}_{\tilde{q}} \circ \Psi = \mathrm{ev}_{\tilde{p}}$ by (1)

$\rightsquigarrow \psi_0 : \widetilde{Y}_{c,\infty} \mapsto \widetilde{X}_{c,\infty}$ such that $\tilde{q} \mapsto \tilde{p}$

$\rightsquigarrow$ extends to $\psi \in \mathcal{G}_\infty[Y, X]$ invertible such that

$$\begin{aligned} \Psi(u) &= u \circ \psi \qquad && \forall u \in \mathcal{G}_\infty(X) \\ \Psi^{-1}(v) &= v \circ \psi^{-1} && \forall v \in \mathcal{G}_\infty(Y). \end{aligned}$$

Sketch of proof for (1)

(1) Non-zero algebra homomorphisms $\varphi : \mathcal{G}_\infty(X) \to \widetilde{\mathbb{K}}_\infty$

Show that they can be identified with elements in $\widetilde{X}_{c,\infty}$:

$$\forall\, \varphi\, \exists!\, \tilde{p} \in \widetilde{X}_{c,\infty} :\ \varphi = \mathrm{ev}_{\tilde{p}}$$

Need to identify the ideals $\ker \varphi \lhd \mathcal{G}_\infty(X)$ with compactly supported generalized points $\widetilde{X}_{c,\infty}$. More precisely look at:

- $\ker \varphi \lhd \mathcal{G}_\infty(X)$:
  - not a maximal ideal
  - but **maximal w.r.t. the property** $\ker \varphi \cap \widetilde{\mathbb{K}}_\infty 1 = \{0\}$
- invertibility in $\widetilde{\mathbb{K}}_\infty$:
  - not every element is invertible
  - but every element is so-called **invertible w.r.t. a characteristic set** S

S-Invertibility: $S \subseteq (0,1]$, $0 \in \overline{S}$

- in $\widetilde{\mathbb{K}}$ and $\mathcal{G}(X)$:
  1. u S-invertible means $\exists v : uv = e_S$ (idempotents in $\widetilde{\mathbb{K}}$)
  2. u is S-invertible $\iff u \cdot e_S + e_{S^c}$ is invertible

- in $\widetilde{\mathbb{K}}_\infty$ and $\mathcal{G}_\infty(X)$ ($= \mathcal{A}$):
  1. there are *no idempotents* in $\widetilde{\mathbb{K}}_\infty$ and $\mathcal{G}_\infty(X)$ (if X connected)

Definition

$u \in \mathcal{A}$ is called S-**invertible** iff $\exists v \in \mathcal{A}\ \exists r \in \widetilde{\mathbb{K}}_\infty$ such that

$$r|_S = 1 \text{ in } \widetilde{\mathbb{K}}_\infty \quad \text{and} \quad uv = r1 \text{ in } \mathcal{A}.$$

Lemma

u is S-invertible $\iff$ u is S-**strictly non-zero**

- in $\widetilde{\mathbb{K}}$ and $\widetilde{\mathbb{K}}_\infty$: every non-zero element is S-invertible

Sketch of proof for (1)

Ideals $\mathcal{J} \lhd \mathcal{G}_\infty(X)$ vs. S-invertibility

$$\mathcal{J} \cap \widetilde{\mathbb{K}}_\infty 1 = \{0\}$$
$$\Updownarrow$$
$\mathcal{J}$ does not contain S-invertibe elements

Characterization of S-invertibility

u is S-invertible in $\mathcal{G}_\infty(X) \iff \forall\, \tilde{p} \in \widetilde{X}_{c,\infty}$: $u(\tilde{p})$ S-invertible in $\widetilde{\mathbb{K}}_\infty$

Applying these results to $\ker \varphi$, finally (1) is obtained:

(1) Non-zero algebra homomorphisms $\varphi : \mathcal{G}_\infty(X) \to \widetilde{\mathbb{K}}_\infty$

$$\forall\, \varphi\, \exists!\, \tilde{p} \in \widetilde{X}_{c,\infty} : \ \varphi = \mathrm{ev}_{\tilde{p}}$$

Where the smooth parametrization jumped in ...

Comparison of invertibility:

- *no idempotents* in $\widetilde{\mathbb{K}}_\infty$ and $\mathcal{G}_\infty(X)$ (if X connected)
- but S-*invertibility* (and S-strictly non-zero) is still applicable

Geometric stability:

- on submanifolds: every generalized function in $\mathcal{G}_\infty(Y)^n$ that is c-bounded into X is already in $\mathcal{G}_\infty[Y, X]$, whereas such elements in $\mathcal{G}(Y)^n$ are only in $\mathcal{G}_{\mathrm{ld}}[Y, X]$ $\rightsquigarrow$ simplifies the result

Further algebraic properties of $\widetilde{\mathbb{K}}_\infty$

As in $\widetilde{\mathbb{K}}$ we have:

- $\widetilde{\mathbb{K}}_\infty$ is a **Gelfand ring**:
  Every prime ideal is contained in a unique maximal ideal.
- $\widetilde{\mathbb{R}}_\infty$ is a **partially ordered ring**
- Properties of **non-invertible elements**:
  - r non-invertible $\Longleftrightarrow$ r zero divisor
  - $rs = 0 \Longrightarrow \exists$ characteristic set S s.t. $r|_S = s|_S = 0$
  - $rs = 0 \Longleftrightarrow \mathrm{Ann}(r) + \mathrm{Ann}(s) = \widetilde{\mathbb{K}}_\infty$
- ...

Other aspects: continuous parametrization

Obviously, $\widetilde{\mathbb{K}}_\infty \hookrightarrow \widetilde{\mathbb{K}}_0 \hookrightarrow \widetilde{\mathbb{K}}$, but moreover:

Theorem

$$\widetilde{\mathbb{K}}_\infty \cong \widetilde{\mathbb{K}}_0 \not\cong \widetilde{\mathbb{K}}$$

Applications of this isomorphism

- $|\_|$, min and max well-defined in $\widetilde{\mathbb{R}}_0 \leadsto$ hence in $\widetilde{\mathbb{R}}_\infty$
- $\widetilde{\mathbb{R}}_\infty$ is a **lattice** (= poset, in which any two elements have a unique sup and inf), although $\mathcal{C}^\infty(0,1]$ is not!

Further investigations ...

$\mathcal{G}_\infty(X) \hookrightarrow \mathcal{G}_0(X) \hookrightarrow \mathcal{G}(X)$, but $\mathcal{G}_\infty(X) \cong \mathcal{G}_0(X)$ as well?