On the formation of trapped surfaces in Einstein-Euler spacetimes

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Outline

Einstein equations

Graviation is a geometric property of space and time.

Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} = 8\pi\, T_{\alpha\beta}$$

$(\mathcal{M}, g_{\alpha\beta})$	...	4-dim. Lorentzian manifold $\rightsquigarrow$ `spacetime`
$R_{\alpha\beta}$	...	Ricci curvature tensor
R	...	scalar curvature
$T_{\alpha\beta}$	...	energy-momentum tensor

Local and global matters

Local existence and uniqueness (1950s)
elliptic initial data constraints + hyperbolic evolution

Example: Scalar field model (Christodoulou 1990s)
$T_{\alpha\beta} = \partial_\alpha\phi\,\partial_\beta\phi - \frac{1}{2}\partial^\nu\phi\,\partial_\nu\phi\,g_{\alpha\beta}$, wave equation for ϕ

- regular solution `dispersing` to infinity for small data
- formation of `trapped surfaces` for certain initial data
- `naked singularities` occur but are instable

Gravitational collapse

Goal

understand the **formation of black holes and singularities**

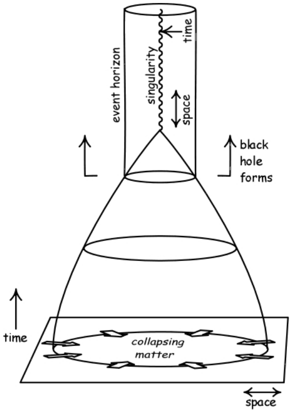
Formation of trapped surfaces for:

Scalar field (Christodoulou 1991)

Collisionless gas (Rendall 1992; Andréasson, Rein 2010)

Perfect fluids (B., LeFloch 2014)

Vacuum (Christodoulou 2008; Klainerman–Rodnianski–Luk 2014)



Formation of a black hole (Norton 2010)

Important notion

Definition

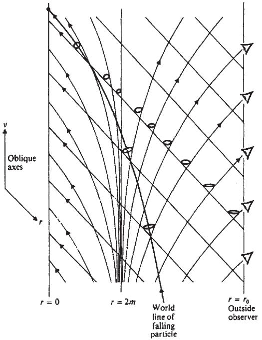
A **trapped surface** is a spacelike surface with `decreasing area` in direction of the future-directed null normals.

Example: Schwarzschild solution

$g = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2 g_{\mathbb{S}^2}$

- $r < 2m \ldots$ trapped surface
- $r > 2m \ldots$ untrapped surface

Via the singularity theorems, the existence of closed trapped surfaces is linked to `geodesic incompleteness` of solutions!



Schwarzschild solution

Outline

Compressible matter

Energy-momentum tensor

$$T^{\alpha\beta} = (\mu + p)u^{\alpha}u^{\beta} + p\,g^{\alpha\beta}$$

μ	...	mass-energy density
$p(\mu) = k^2\mu$	...	pressure with $k \in (0, 1)$ speed of sound
u^{α}	...	velocity vector, normalized to $u^{\alpha}u_{\alpha} = -1$
$\nabla_{\alpha}T^{\alpha\beta} = 0$	...	Euler equations

Einstein–Euler equations

$$R_{\alpha\beta} - \tfrac{1}{2}R\,g_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad \text{and} \quad \nabla_{\alpha}T^{\alpha\beta} = 0$$

We consider spacetimes that satisfy the Einstein–Euler equations in the **distributional sense.**

Geometric and fluid components – 4 unknowns

We assume **spherical symmetry** and thus use ...

Generalized Eddington–Finkelstein coordinates

$$g = -ab^2 dv^2 + 2b\,dv\,dr + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2)$$

Geometry

a, b

- $b(v,r) > 0$, $a(v,r)$ may change sign
- $a < 0$: trapped, $a > 0$: untrapped
- $\lim_{r\to 0} a(v,r) = \lim_{r\to 0} b(v,r) = 1$

Normalized fluid variables

$$M = b^2 \mu u^0 u^0 \quad \text{and} \quad V = \frac{u^1}{bu^0} - \frac{a}{2}$$

Fluid

M, V

- M positive, V negative

Reduced Einstein–Euler system

Rewrite principal part in divergence form using:

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = M \begin{pmatrix} 1 \\ \frac{a}{2} + \frac{1-k^2}{1+k^2} V \end{pmatrix}$$

Proposition (B., LeFloch 2014)

*The full Einstein–Euler system in spherically symmetry is equivalent to a **system of two balance laws** for the fluid,*

$$\partial_v U + \partial_r F(U, a, b) = S(U, a, b),$$

*and **two integral equations** for the geometry,*

$$b(v, r) = e^{4\pi(1+k^2) \int_0^r M(v,s) s \, ds},$$

$$a(v, r) = 1 - \frac{4\pi(1+k^2)}{r} \int_0^r \frac{b(v,s)}{b(v,r)} M(v, s)(1 - 2\frac{1-k^2}{1+k^2} V(v, s)) s^2 \, ds.$$

Main result (summary)

Theorem (B., LeFloch 2014)

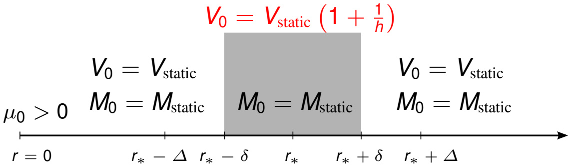
There exists a class of `untrapped initial data` *prescribed on a hypersurface, that evolves to spherically symmetric Einstein–Euler spacetimes with bounded variation that contain* `trapped surfaces`*.*

Tasks:

- deal with low regularity
- find admissible initial data
- solve initial value problem and estimate time of existence
- show that trapping occurs during time of existence

1. Initial data

1. use the smooth **static solution** $(M, V, a, b)_{\text{static}}$ for an initial value $\mu_0 > 0$ (Rendall, Schmidt 1991) and
2. add a **compact fluid perturbation** around r_* with parameters h, δ

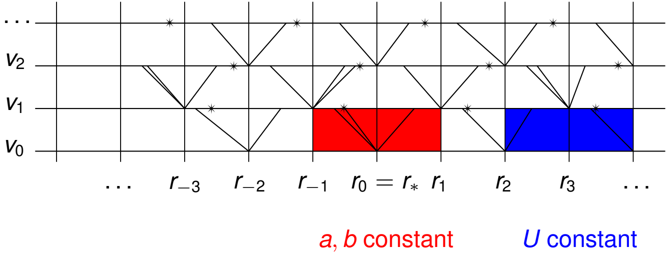


$\Longrightarrow$ yields **admissible (untrapped) initial data** (M_0, V_0, a_0, b_0) for certain ratios of δ/h

2. Local existence result for the initial value problem

is based on a **generalized random choice scheme** on a discretized grid (Glimm 1972; Groah, Temple 2004 etc.)

1. solve $\partial_v U + \partial_r F(U) = 0$ on uniform background
2. evolve solution using $\partial_v W = S(\text{ource}) + G(\text{eometry})$



$\Longrightarrow$ yields **approximate solutions** $(M_\sharp, V_\sharp, a_\sharp, b_\sharp)$

3. Evolution of perturbation

- control growth of $M_\sharp, V_\sharp, a_\sharp, b_\sharp$ in terms of h, δ and $\kappa = 1 + \frac{8k(1+k^2)}{(1-k^2)^2}$:

$$\frac{1}{C_0} e^{-C\frac{v-v_0}{h^\kappa}} \leq M_\sharp \leq C_0 e^{C\frac{v-v_0}{h^\kappa}}$$

$$\frac{1}{C_0} e^{-C\frac{v-v_0}{h^\kappa}} \left(1 + \frac{1}{h}\right) \leq -V_\sharp \leq C_0 e^{C\frac{v-v_0}{h^\kappa}} \left(1 + \frac{1}{h}\right)$$

$$-\frac{1}{h} \leq a_\sharp \leq 1 \leq b_\sharp \leq C_b$$

- estimate domain of dependence Ω_*
- estimate minimal time of existence $v_* = v_0 + \tau h^\kappa$

$\Longrightarrow$ **initial perturbation is preserved during evolution**

Existence result

The `approximate solutions` satisfy

- certain growth bounds related to the initial perturbation,
- weak regularity $L^\infty_v(BV_r) \cap Lip_v(L^1_r)$

Consequently, a subsequence converges pointwise:

Theorem (B., LeFloch 2014)

Given `admissible initial data` *(M_0, V_0, a_0, b_0) at v_0, there exists a* ***bounded variation solution to the Einstein–Euler equations in spherical symmetry*** *up to time $v_* = v_0 + \tau h^\kappa$ which itself satisfies these growth bounds.*

Formation of trapped surfaces

A careful analysis of the speed of propagation and growth of the solution yields:

Corollary (B., LeFloch 2014)

Fix $k \in (0,1)$ *such that* $\kappa < 2$*. Suppose the initial data with* $\frac{\delta}{h} = \frac{1}{C_1}$ *additionally satisfy*

$$8\pi r_* > e^3 \Lambda C_0^3 C_1 .$$

Then, if h is chosen sufficiently small, a ***trapped surface*** *forms in the solution before time* v_**, i.e. there exists* $(v, r) \in \Omega_*$ *such that* $a(v, r) < 0$*.*

Thank you for your attention!